JOURNAL OF APPROXIMATION THEORY 8, 142–149 (1973)

Hermite–Birkhoff Interpolation Problems in Haar Subspaces

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Communicated by E. W. Cheney

Received March 15, 1971

DEDICATED TO PROFESSOR I. J. SCHOENBERG
ON THE OCCASION OF HIS 70TH BIRTHDAY

1. INTRODUCTION

Let $E = (E_{ij})$, $i = 1,..., p, j = 0, 1,..., q$, be a rectangular matrix with p rows ($p \geq 1$) and $q + 1$ columns ($q \geq 0$) such that each element of it is either 0 or 1. Such a matrix is called an *incidence matrix*. Let n be the total number of 1's in E,

$$n = \sum_{i=1}^{p} \sum_{j=0}^{q} E_{ij}.$$

In the sequel we assume $1 \leq q \leq n - 1$.

Let H be an n-dimensional subspace of $C^q(a, b)$, the space of q-times continuously differentiable real functions defined on an open interval (a, b). Select arbitrarily a set of p nodes, $x_1 < x_2 < \cdots < x_p$, from (a, b). Further, corresponding to each (i, j) for which $E_{ij} = 1$, select an arbitrary number $a_i^{(j)}$.

We consider the following *Hermite–Birkhoff interpolation problem*. Find a function $h \in H$ which satisfies the n interpolatory conditions:

$$h^{(j)}(x_i) = \frac{d^j h}{dx^j}(x_i) = a_i^{(j)}, \tag{1}$$

$$\text{whenever} \quad E_{ij} = 1: \quad i = 1, 2,..., p, \quad j = 0, 1,..., q.$$

We shall say that E is *poised with respect to H on (a, b)* if there always exists a unique solution to the above problem, regardless of the choice of the nodes $x_1 ,..., x_p$ and the numbers $a_i^{(j)}$.

* Work on this paper was supported in part by Grant DA-ARO(D)-31-124-G1050, Army Research Office, Durham, and by National Science Foundation Grant GP-23655 awarded to The University of Texas at Austin.

We now mention a number of relevant papers related to our main result, which is given in the next section. G. Pólya [5], I. J. Schoenberg [7], and K. Atkinson and A. Sharma [1] consider the case where $H = \pi_{n-1}$, the space of all polynomials of degree at most $n - 1$. The results obtained by the latter authors contain ours for this special case. I. J. Schoenberg assumes a Hermite-type condition on E, that is, for $i = 1,..., p$, $E_{i,j+1} = 1$ implies $E_{i,j} = 1$. On the other hand, J. W. Matthews [4] proves our main theorem for the case in which E consists of only two columns ($q = 1$) and satisfies the above Hermite-type condition. W. Haussmann [2], generalizing Matthews' result but still assuming the Hermite-type condition on E, provides a simpler proof for Matthews' result and proves further that, if

$$\dim H = \dim H^{(1)} = n(H^{(1)} = \{dh/dx : h \in H\}),$$

and both H and $H^{(1)}$ are Haar subspaces, then E is poised with respect to H on (a, b). Our main theorem does not include this last fact, even though both proofs are based on the same known theorem quoted in Sec. 3.

2. Main Theorem

Let H be an n-dimensional subspace of $C^q(a, b)$ and satisfy the condition

$$\dim H^{(q)} = \dim\{h^{(q)}(x) : h \in H\} = n - q \qquad (q \leqslant n - 1)$$

and let $H^{(q)}$ satisfy the Haar condition on (a, b), i.e., every nonzero function in $H^{(q)}$ vanishes at $\dim H^{(q)} - 1$ or less distinct points on (a, b). Then, in order that E be poised with respect to H on (a, b) it is sufficient that E satisfies the *Pólya condition* and is *conservative*.

We say that E satisfies the Pólya condition if

$$M_0 \geqslant 1,\ M_1 \geqslant 2,...,\ M_q \geqslant q + 1,$$

where

$$M_j = m_0 + m_1 + \cdots + m_j \qquad (j = 0, 1,..., q),$$

$$m_j = \sum_{i=1}^{p} E_{ij} = \text{the number of 1's in the } j\text{th column of } E.$$

In order to define the conservativeness of E, let us first define a *sequence* in E as in [1]. A sequence in E is a maximal sequence of 1's in any row of E. For instance, the matrix

$$E = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

has a sequence $\{1, 1\}$ starting at $(1, 1)$-position. It has seven sequences in all. For convenience, if a sequence starts at (i, j)-position, we call it *the* (i, j)-*sequence*. A sequence is *even* or *odd* depending on whether the number of 1's in this sequence is even or odd. We say that E is conservative if an arbitrary sequence (let it be (i_0, j_0)-sequence) is either even or else the upper left portion of E defined by

$$\{E_{ij} : i < i_0 \text{ and } j < j_0\}$$

and the lower left portion of E defined by

$$\{E_{ij} : i > i_0 \text{ and } j < j_0\}$$

do not simultaneously contain 1's. One or both of these sets may be empty. For instance, the last matrix E is not conservative since the $(3, 2)$-sequence $\{1, 1, 1\}$ is odd and the upper left portion of E and the lower left portion of E defined in the above simultaneously contain 1's (there are two 1's in the former and one 1 in the latter). The matrix

$$E = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

is conservative. Note that if E satisfies the Hermite-type condition mentioned in the last section then E is automatically conservative.

Before proceeding to the proof, we note a few simple consequences of the hypothesis of the main theorem. They are

(a) $\dim H^{(i)} = n - i$ $(i = 0, 1, \ldots, q)$, where $H^{(i)} = \{h^{(i)}(x) : h \in H\}$;
(b) $H^{(0)} = H, H^{(1)}, \ldots, H^{(q)}$ are all Haar subspaces;
(c) H contains polynomials $1, x, \ldots, x^{q-1}$.

To prove (a), it is enough to prove that

$$\dim H^{(i)} - 1 \leq \dim H^{(i+1)} \leq \dim H^{(i)} \; (i = 0, \ldots, q-1).$$

The proof of this last fact is easy and is omitted. To prove (b), use Rolle's theorem and (a). Part (c) follows easily from (a).

3. Proof of the Main Theorem

In order to prove the main theorem, it is necessary and sufficient to prove that the homogeneous system obtained from (1) has only the trivial solution $h = 0$. Thus let $h \in H$ satisfy (1) with all $a_j^{(i)}$'s 0. In order to prove $h = 0$, it is enough to prove that $h^{(q)} = 0$ and $h^{(i)}$ vanishes on (a, b) at least once for $i = 0, 1, ..., q - 1$. In order to prove $h^{(q)} = 0$, it is enough to prove that $h^{(q)}$ vanishes at least $n - q (= \dim H^{(q)})$ times on (a, b), counting each *double zero* twice, on the strength of the following theorem.

THEOREM. ([3, *Theorem* 4.2, *p.* 23] *and* [6, *Lemma* 3-2, *p.* 57]) *If M is an m-dimensional Haar subspace of $C[a, b]$ and if a function f in M vanishes on $[a, b]$ at least m times, counting each double zero twice, then $f = 0$ on $[a, b]$.*

An interior point c of $[a, b]$ is called a *simple zero* (resp., a double zero) of $f \in C[a, b]$ if $f(c) = 0$ and if f changes (resp., does not change) sign at c in the sense that

$$f(t_1)f(t_2) \leqslant 0 \text{ (resp., } f(t_1)f(t_2) > 0)$$

whenever $t_1 \in (c - \delta, c)$ and $t_2 \in (c, c + \delta)$ with a sufficiently small but fixed $\delta > 0$. If c is an endpoint ($c = a$ or $c = b$) and $f(c) = 0$, then c is a simple zero by definition.

Let us write E_h for E to emphasize the fact that the incidence matrix E is associated with the function h. In the sequel we shall construct, by induction, a sequence of incidence matrices $E^{(0)}_{h^{(0)}} = E_h$, $E^{(1)}_{h^{(1)}}, ..., E^{(q)}_{h^{(q)}}$, where $E^{(i)}_{h^{(i)}}$ denotes an incidence matrix associated with $h^{(i)}$, $i = 0, 1, ..., q$, in such a way that the following properties are satisfied:

(i) $E^{(i)}_{h^{(i)}}$ has one less number of columns than $E^{(i-1)}_{h^{(i-1)}}$, $i = 1, ..., q$;

(ii) $E^{(i)}_{h^{(i)}}$ has at least $n - i$ 1's, *where we count each* 1 *in its right-most column* (*and only in its right-most column*) *twice if that* 1 *indicates a double zero of* $h^{(q)}$.

(iii) $E^{(i)}_{h^{(i)}}$ satisfies the Pólya condition and is conservative, $i = 0, ..., q$.

If the indicated sequence of matrices is in fact constructed, then, by (i), the right-most column of $E^{(i)}_{h^{(i)}}$ $(i = 0, ..., q)$ always corresponds to the function $h^{(q)}$ and the matrix $E^{(q)}_{h^{(q)}}$ consists of a single column. Then, by (ii), $h^{(q)}$ vanishes at least $n - q$ times, counting each double zero twice. Property (iii) implies that $h^{(i)}$, $i = 0, ..., q - 1$, vanishes at least once on (a, b). Thus, the proof will be complete if we construct the indicated sequence $\{E^{(i)}_{h^{(i)}}\}$ satisfying properties (i)–(iii) in the above.

The incidence matrix $E^{(0)}_{h^{(0)}} = E_h$ satisfies the indicated properties by definition, where property (i) is satisfied vacuously. Let

$$X_j = \{x_i : E_{i,j} = 1\}, \qquad j = 0, 1, ..., q;$$

that is,

$$h^{(j)}(x_i) = 0 \qquad \text{for} \qquad x_i \in X_j .$$

By the Pólya condition, the left-most column of E_h contains at least one 1. Hence X_0 is not empty. If E_h contains exactly one 1, let $E_{h^{(1)}}^{(1)}$ be defined to be that matrix which is obtained from E_h by deleting its left-most column. Then, $E_{h^{(1)}}^{(1)}$ clearly satisfies properties (i)-(iii). Thus, suppose that E_h contains 2 or more 1's in its left-most column. Let X_0 contain k points, $t_1 < t_2 < \cdots < t_k$. Hence $h(t_1) = \cdots = h(t_k) = 0$. Take the interval $[t_1, t_2]$ and let $|h|$ take the maximum on it at ξ_1,

$$|h(\xi_1)| = \max_{[t_1, t_2]} |h|.$$

We may assume $t_1 < \xi_1 < t_2$. Obviously, $h^{(1)}(\xi_1) = 0$. Here two separate cases may arise: (a) $\xi_1 \notin X_1$ and (b) $\xi_1 \in X_1$.

If (a) is the case, then the fact that $h^{(1)}(\xi_1) = 0$ is new information about $h^{(1)}$ which is not indicated by E_h. For instance, if

$$E_h = \begin{array}{c}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}\end{array} \begin{array}{l} \cdots x_1 \\ \cdots x_2, \\ \cdots x_3 \end{array}$$

then $X_0 = \{x_1, x_3\}$ and $X_1 = \{x_1\}$. Since $x_1 < \xi_1 < x_3$ ($|h(\xi_1)| = \max_{[x_1, x_3]} |h|$), $\xi_1 \notin X_1$.

If (b) is the case, $\xi_1 = x_{i_0} \in X_1$ for some i_0. By the hypotheses that $t_1 < t_2 < \cdots < t_k$ is the complete enumeration of the nodes appearing in X_0 and by the fact that $t_1 < \xi_1 < t_2$, the node x_{i_0} does not appear in X_0. This means the existence of a $(i_0, 1)$-sequence in E_h. For instance, if

$$E_h = \begin{array}{c}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\end{array} \begin{array}{l} \cdots x_1, \\ \cdots x_2, \\ \cdots x_3 \end{array}$$

then $X_0 = \{x_1, x_3\}$ and $X_1 = \{x_1, x_2\}$. It may happen that $\xi_1 = x_2$ $(i_0 = 2)$. Now, since E_h is conservative, the last $(i_0, 1)$-sequence must be even. Let this sequence consist of m 1's (m, even). Then

$$h^{(1)}(\xi_1) = h^{(2)}(\xi_1) = \cdots = h^{(m)}(\xi_1) = 0.$$

From the fact that h takes a local extremum at ξ_1, it follows that ξ_1 is a *simple* zero of $h^{(1)}$. It then follows that ξ_1 is a double zero of $h^{(2)}, h^{(4)}, \ldots, h^{(m)}$ and is a simple zero of $h^{(3)}, \ldots, h^{(m-1)}$ (we omit the easy proof of this fact). Since ξ_1 is a *double* zero of $h^{(m)}$, it is a simple zero of $h^{(m+1)}$ provided that $h^{(m)} \in C^1$. In other words, if the $(i_0, 1)$-sequence under consideration terminates before the right-most column of E_h, we can write an additional 1 at the $(i_0, m+1)$-position of E_h. Otherwise we have $m = q$ and ξ_1 is, therefore, a double zero of $h^{(q)}$.

We can carry out the same analysis for the rest of the subintervals $[t_2, t_3],..., [t_{k-1}, t_k]$. Let $\xi_2 ,..., \xi_{k-1}$ play similar roles as ξ_1, i.e., $t_2 < \xi_2 < t_3$, $h(\xi_2) = \max_{[t_2,t_3]} | h |$, etc. Using these $\xi_1 ,..., \xi_{k-1}$, we can construct an incidence matrix $E_{h^{(1)}}^{(1)}$ for $h^{(1)}$ in an obvious way. For instance, if

$$E_h = \begin{array}{c@{}l} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \cdots x_1 \\ \cdots x_2 \\ \cdots x_3 \\ \cdots x_4 \end{array} \end{array},$$

then $X_0 = \{x_1, x_3, x_4\}$ $(t_1 = x_1, t_2 = x_3, t_3 = x_4)$ and $X_1 = \{x_2\}$. The matrix E_h is conservative and satisfies the Pólya condition. Thus, if $\xi_1 \neq x_2$, we take $E_{h^{(1)}}^{(1)}$ to be

$$E_{h^{(1)}}^{(1)} = \begin{array}{c@{}l} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \cdots x_2 \\ \cdots \xi_1 \\ \cdots x_3 \\ \cdots \xi_2 \\ \cdots x_4 \end{array} \end{array},$$

where $x_1 < x_2 < \xi_1 < x_3$ (here, of course, the case $x_1 < \xi_1 < x_2 < x_3$ may occur, in which case we can construct $E_{h^{(1)}}^{(1)}$ similarly). If $\xi_1 = x_2$, then

$$E_{h^{(1)}}^{(1)} = \begin{array}{c@{}l} \begin{bmatrix} 1 & 1 & 1^* \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} \cdots x_2 \\ \cdots x_3 \\ \cdots \xi_2 \\ \cdots x_4 \end{array} \end{array},$$

where $\xi_1 = x_2$ and the 1 with an asterisk (*) is new information $(h^{(3)}(x_2) = 0)$.

For yet another case, take

$$E_h = \begin{array}{c@{}l} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \begin{array}{l} \cdots x_1 \\ \cdots x_2 \\ \cdots x_3 \end{array} \end{array}.$$

If $\xi_1 = x_2$ $(x_1 < \xi_1 < x_3)$, then we take $E_{h^{(1)}}^{(1)}$ to be

$$E_{h^{(1)}}^{(1)} = [1 \;\; 1^*] \cdots x_2,$$

where the 1 with an asterisk (*) indicates that x_2 is a double zero of $h^{(2)}$.

From the construction of $E_{h^{(1)}}^{(1)}$, we can verify without difficulty that conditions (i)–(iii) are always satisfied for $i = 1$. In fact, verification of (i) is trivial. In order to verify (ii) we note that for each pair of consecutive 1's in the left-most column of E_h, a new entry 1 is introduced in $E_{h^{(1)}}^{(1)}$ in addition to those 1's in $E_{h^{(1)}}^{(1)}$ which are already present in E_h. In order to verify the conservativeness of $E_{h^{(1)}}^{(1)}$ we need only to check those sequences in $E_{h^{(1)}}^{(1)}$ which do not originate in the left-most column of $E_{h^{(1)}}^{(1)}$. Each of these sequences is, therefore, already present in E_h, except perhaps its last term. From this consideration it is now clear that $E_{h^{(1)}}^{(1)}$ is conservative. There remains only to show that $E_{h^{(1)}}^{(1)}$ satisfies the Pólya condition. Suppose the contrary. Then there would exist a j_0 such that

$$\begin{aligned} &m_0(E_{h^{(1)}}^{(1)}) \geq 1 \\ &\vdots \\ &m_0(E_{h^{(1)}}^{(1)}) + m_1(E_{h^{(1)}}^{(1)}) + \cdots + m_{j_0-1}(E_{h^{(1)}}^{(1)}) \geq j_0 \end{aligned}$$

but

$$m_0(E_{h^{(1)}}^{(1)}) + \cdots + m_{j_0}(E_{h^{(1)}}^{(1)}) < j_0 + 1,$$

where $m_j(E_{h^{(1)}}^{(1)})$ denotes the number of 1's in the j-th column of $E_{h^{(1)}}^{(1)}$, $j = 0, 1, \ldots, q - 1$. This means that the number of the 1's in $E_{h^{(1)}}^{(1)}$ contained to the left and including the j_0-th column of $E_{h^{(1)}}^{(1)}$ is exactly j_0 and that the j_0-th column contains no 1's. Hence the newly introduced entries in $E_{h^{(1)}}^{(1)}$ (i.e., those 1's which correspond to $\xi_1, \xi_2, \ldots$) are to be found in $E_{h^{(1)}}^{(1)}$ to the left of the j_0-th column. But, from the construction, the number of the 1's contained in the same area of $E_{h^{(1)}}^{(1)}$ is given by

$$\{m_0(E_h) - 1\} + m_1(E_h) + \cdots + m_{j_0-1}(E_h) \geq j_0 + 2 - 1 = j_0 + 1$$

which is a contradiction.

Starting with $E_{h^{(1)}}^{(1)}$, we can similarly construct $E_{h^{(2)}}^{(2)}, \ldots, E_{h^{(q)}}^{(q)}$ which all satisfy conditions (i)–(iii). This completes the proof.

4. Remark

Under the hypothesis of the main theorem we can prove that the Pólya condition is necessary in order that E is poised with respect to H on (a, b). The proof runs in a similar line as in [7, p. 540]. In fact, if the Pólya condition is not satisfied then there is a j_0 such that $m_0 + m_1 + \cdots + m_{j_0} < j_0 + 1$ with $j_0 < q$. Since π_{q-1} is contained in H (see Sec. 2), so is π_{j_0}. It is possible to find a nontrivial polynomial $h(x) = c_0 + c_1 x + \cdots + c_{j_0} x^{j_0}$ which satisfies the homogeneous system (1), because, for $j > j_0$ $h^{(j)} = 0$ is automatically satisfied and for $j \leqslant j_0$ the number $j_0 + 1$ of unknowns $(c_0, \ldots, c_{j_0})$ exceeds the number $m_0 + \cdots + m_{j_0}$ of equations.

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